

# THE GAUGE TRANSFORMATION OF THE CONSTRAINED SEMI-DISCRETE KP HIERARCHY

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**ABSTRACT.** In this paper, the gauge transformation of the constrained semi-discrete KP(cdKP) hierarchy is constructed explicitly by the suitable choice of the generating functions. Under the  $m$ -step successive gauge transformation  $T_m$ , we give the transformed (adjoint) eigenfunctions and the  $\tau$ -function of the transformed Lax operator of the cdKP hierarchy.

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## 1. INTRODUCTION

The semi-discrete Kadomtsev-Petviashvili (dKP) hierarchy [1, 2, 3, 4, 5, 6] is an attractive research object in the field of the discrete integrable systems. The dKP hierarchy is defined by means of the difference derivative  $\Delta$  instead of the usual derivative  $\partial$  with respect of  $x$  in a classical system [7, 8], and the continuous spatial variable is replaced by a discrete variable  $n$ . By using a non-uniform shift of space variable, the  $\tau$ -function of KP hierarchy implies a special kind of  $\tau$ -function for the semi-discrete KP hierarchy [2]. The ghost symmetry of the dKP hierarchy is constructed by using the additional symmetry [4]. The dKP hierarchy possesses an infinite dimensional algebra structure [5]. Very recently, the continuum limit of the dKP hierarchy is given in ref. [6].

Gauge transformation is one kind of powerful method to construct the solutions of the integrable systems for both the continuous KP hierarchy [9, 10, 11, 12, 13, 14] and the dKP hierarchy [15, 16]. The multi-fold of this transformation is expressed directly by determinants [14, 16], and is used to construct multiple wave solutions to the generalized KP and BKP equations [17]. This transformation is also applicable to the so-called constrained KP hierarchy [18, 19, 20, 21, 22]. It is known that there are two types of gauge transformation: differential type  $T_d$  and integral type  $T_i$ . Because of the reduction conditions of the BKP hierarchy and the CKP hierarchy, it is necessary to consider the pair of  $T_d$  and  $T_i$ . This has been used to construct the two-peak soliton [23] and to construct the gauge transformation of the constrained BKP

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hierarchy and the constrained CKP hierarchy [24]. And it is known that the KP hierarchy has been generalized to constrained flows and extended flows with self-consistent sources [25]. It is interesting to note that the determinant representation of the dressing transformation of the extended two-dimensional Toda lattice hierarchy is also developed [26].

Similar to the constrained KP hierarchy [27, 28, 29], the constrained semi-discrete KP(cdKP) hierarchy [30] is defined by  $\frac{\partial L}{\partial t_l} = [(L^l)_+, L]$  with a Lax operator  $L = \Delta + \sum_{i=1}^m q_i(t)\Delta^{-1}r_i(t)$ . The so called discrete non-linear Schrödinger (generalized DNLS) equation [31]

$$\begin{aligned} q_{1,t_2} &= \Delta^2 q_1 + 2q_1^2 r_1, \\ r_{1,t_2} &= -\Delta^{*2} r_1 + 2q_1 r_1^2, \end{aligned}$$

can be generated from the  $t_2$  flows of cdKP hierarchy. Further the additional symmetry of the cdKP hierarchy is constructed in ref. [30], which shows that the constraint in the dKP hierarchy preserves the symmetry structures with very minor modification. However the gauge transformation of the cdKP hierarchy has not appeared in literatures.

The purpose of this paper is to construct the gauge transformation of the cdKP hierarchy. As we shall show, it is not a trivial task to reduce the gauge transformation of the dKP hierarchy to the cdKP hierarchy. For the cdKP hierarchy, the transformed eigenfunctions and the adjoint eigenfunctions can not be conserved for the originally form. If the generating functions of gauge transformation of the cdKP hierarchy are selected from the (adjoint)eigenfunctions, we get the  $\Delta$ -Wronskian representation of the transformed  $\tau$  function.

This paper is organized as follows. Some basic results of the dKP hierarchy and the cdKP hierarchy are summarized in Section 2. After introducing of two types gauge transformations of the cdKP hierarchy, the transformation rules of the eigenfunction functions and the  $\tau$  function of the cdKP hierarchy are obtained by means of a crucial modification from the transformation of the dKP hierarchy in Section 3. Next the successive applications of the difference type gauge transformation have be discussed in Section 4. And we establish the determinant representation of gauge transformation operator  $T_m$ , then obtained a general form of the  $\tau$ -function  $\tau_{\Delta}^{(m)}$  for the cdKP hierarchy. Section 5 is devoted to conclusions and discussions.

## 2. THE CONSTRAINED SEMI-DISCRETE KP HIERARCHY

Let us briefly recall some basic facts about the semi-discrete KP (cdKP) hierarchy according to reference [2]. Firstly a space  $F$ , namely

$$F = \{f(n) = f(n, t_1, t_2, \dots, t_j, \dots); n \in \mathbb{Z}, t_i \in \mathbb{R}\} \quad (2.1)$$

is defined for the space of the semi-discrete KP hierarchy.  $\Lambda$  and  $\Delta$  are denote for the shift operator and the difference operator, respectively. Their actions on function  $f(n)$  are defined as

$$\Lambda f(n) = f(n+1) \quad (2.2)$$

and

$$\Delta f(n) = f(n+1) - f(n) = (\Lambda - I)f(n) \quad (2.3)$$

respectively, where  $I$  is the identity operator.

For any  $j \in \mathbb{Z}$ , the Leibniz rule of  $\Delta$  operation is,

$$\Delta^j \circ f = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^i f)(n+j-i) \Delta^{j-i}, \quad \binom{j}{i} = \frac{j(j-1) \cdots (j-i+1)}{i!}. \quad (2.4)$$

So an associative ring  $F(\Delta)$  of formal pseudo difference operators is obtained, with the operation “+” and “ $\circ$ ”, namely  $F(\Delta) = \left\{ R = \sum_{j=-\infty}^d f_j(n) \Delta^j, f_j(n) \in R, n \in \mathbb{Z} \right\}$ . The adjoint operator to the  $\Delta$  operator is given by  $\Delta^*$ ,

$$\Delta^* \circ f(n) = (\Lambda^{-1} - I)f(n) = f(n-1) - f(n), \quad (2.5)$$

where  $\Lambda^{-1}f(n) = f(n-1)$ , and the corresponding “ $\circ$ ” operation is

$$\Delta^{*j} \circ f = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^{*i} f)(n+i-j) \Delta^{*j-i}. \quad (2.6)$$

Then the adjoint ring  $F(\Delta^*)$  to the  $F(\Delta)$  is obtained, and the formal adjoint to  $R \in F(\Delta)$  is defined by  $R^* \in F(\Delta^*)$  as  $R^* = \sum_{j=-\infty}^d \Delta^{*j} \circ f_j(n)$ . The “ $*$ ” operation satisfies the rules as  $(F \circ G)^* = G^* \circ F^*$  for two operators  $F$  and  $G$  and  $f(n)^* = f(n)$  for a function  $f(n)$ .

We list some useful properties for the difference operators as following:

**Lemma 2.1.** *For  $f \in F$ ,  $\Delta$  and  $\Lambda$  as above, the following identities hold.*

$$(1) \quad \Delta \circ \Lambda = \Lambda \circ \Delta, \quad (2.7)$$

$$(2) \quad \Delta^* = -\Delta \circ \Lambda^{-1}, \quad (2.8)$$

$$(3) \quad (\Delta^{-1})^* = (\Delta^*)^{-1} = -\Lambda \circ \Delta^{-1}, \quad (2.9)$$

$$(4) \quad \Delta^{-1} \circ f \circ \Delta^{-1} = (\Delta^{-1} f) \circ \Delta^{-1} - \Delta^{-1} \circ \Lambda(\Delta^{-1} f), \quad (2.10)$$

$$(5) \quad \Delta \circ f(n) = \Lambda(f(n)) \circ \Delta + \Delta(f(n)). \quad (2.11)$$

The so-called 1-constrained semi-discrete KP (cdKP) hierarchy [30] is defined by the following Lax equation

$$\frac{\partial L}{\partial t_l} = [(L^l)_+, L], l = 1, 2, \dots, \quad (2.12)$$

associated with a special Lax operator

$$L = L_+ + \sum_{i=1}^m q_i(t) \Delta^{-1} r_i(t) = \Delta + \sum_{i=1}^m q_i(t) \Delta^{-1} r_i(t), \quad (2.13)$$

and  $q_i(t)$  is an eigenfunction,  $r_i(t)$  is an adjoint eigenfunction of the Lax operator  $L$ . The eigenfunction and adjoint eigenfunction  $q_i(t), r_i(t)$  are important dynamical variables in the

cdKP hierarchy. One can check that the Lax equation (2.12) is consistent with the evolution equations of the eigenfunction(or adjoint eigenfunction)

$$\begin{cases} q_{i,t_m} = B_m q_i, \\ r_{i,t_m} = -B_m^* r_i, \quad B_m = (L^m)_+, \forall m \in N. \end{cases} \quad (2.14)$$

Therefore the cdKP hierarchy in eq.(2.12) is well defined. From the Lax equation (2.12), we get the first nontrivial  $t_2$  flow equations of the cdKP hierarchy for  $m = 1, l = 2$  as

$$\begin{cases} q_{1,t_2} = \Delta^2 q_1 + 2q_1^2 r_1 = q_1(n+2) - 2q_1(n+1) + q_1(n) + 2q_1^2 r_1, \\ r_{1,t_2} = -\Delta^{*2} r_1 + 2q_1 r_1^2 = r_1(n) - 2r_1(n-1) + r_1(n-2) + 2q_1(n)r_1(n)^2 \end{cases} \quad (2.15)$$

And it is so called the generalized discrete non-linear Schrödinger (generalized DNLS) equation [31]. It can be reduced to the discrete non-linear Schrödinger (DNLS) equation [31] by letting  $r_1 = q_1^*$  and a scaling transformation  $t_2 = it_2$ .

### 3. GAUGE TRANSFORMATIONS OF THE CONSTRAINED SEMI-DISCRETE KP HIERARCHY

We will discuss the gauge transformations of the constrained semi-discrete KP hierarchy in this section. It is reported two types of gauge transformation operators for the semi-discrete KP hierarchy in [16]. We will extended the gauge transformation to the constrained semi-discrete KP hierarchy. If there exist a pseudo-difference operator  $T$  satisfying

$$L^{(1)} = T \circ L \circ T^{-1}, B_n^{(1)} = (L^{(1)})_+^n, \quad (3.1)$$

so that

$$\frac{\partial L}{\partial t_l} = [(L^l)_+, L]$$

holds for the transformed Lax operator  $L^{(1)}$ , i.e.,

$$\frac{\partial L^{(1)}}{\partial t_l} = [(L^{(1)})_+^l, L^{(1)}]; \quad (3.2)$$

then  $T$  is called a gauge transformation operator of the cdKP hierarchy. According to the definition of gauge transformation, we have the following criterion lemma.

**Lemma 3.1.** *The operator  $T$  is a gauge transformation operator, if*

$$(T \circ B_n \circ T^{-1})_+ = T \circ B_n \circ T^{-1} + \frac{\partial T}{\partial t_n} \circ T^{-1}, \quad (3.3)$$

or

$$(T \circ B_n \circ T^{-1})_- = -\frac{\partial T}{\partial t_n} \circ T^{-1}. \quad (3.4)$$

Similar to the KP hierarchy and the cKP hierarchy, there are two types of gauge transformation operators of the cdKP hierarchy as the following lemma:

**Lemma 3.2.** [9, 15] *The cdKP hierarchy have two types gauge transformation operators, namely,*

$$(1).T_d(q) = \Lambda(q) \circ \Delta \circ q^{-1}, \quad (3.5)$$

$$(2).T_i(r) = \Lambda^{-1}(r^{-1}) \circ \Delta^{-1} \circ r. \quad (3.6)$$

Where  $q$  and  $r$  are defined by (2.14) that are the (adjoint) eigenfunction of  $L$  in (2.13), which is called the generating functions of gauge transformation.

Via the gauge transformations of two types,  $L^{(0)} = L$  becomes  $L^{(1)}$  by the following lemma.

**Theorem 3.3.** *Under the gauge transformation of  $T_d(q)$ , the transformed Lax operator reads as*

$$L^{(1)} = L_+^{(1)} + L_-^{(1)}, \quad (3.7)$$

$$L_+^{(1)} = \Lambda(L_+^{(0)}) + \Lambda(q) \circ \Delta(q^{-1}L_+^{(0)}q)_{\geq 1} \circ \Delta^{-1} \circ \Lambda(q^{-1}), \quad (3.8)$$

$$L_-^{(1)} = q_0^{(1)}\Delta^{-1}r_0^{(1)} + \sum_{i=1}^m q_i^{(1)}\Delta^{-1}r_i^{(1)}, \quad (3.9)$$

$$q_0^{(1)} = T_d(q)(L^{(0)})(q), r_0^{(1)} = \Lambda(q^{-1}), \quad (3.10)$$

$$q_i^{(1)} = T_d(q)q_i^{(0)}, r_i^{(1)} = (T_d^{-1})^*(q)(r_i^{(0)}). \quad (3.11)$$

*Proof.*

$$\begin{aligned} L_+^{(1)} &= (T_d(q) \circ L^{(0)} \circ T_d^{-1}(q))_+ \\ &= (\Lambda(q) \circ \Delta \circ q^{-1} \circ L_+^{(0)} \circ q \circ \Delta^{-1} \circ \Lambda(q^{-1}))_+ \\ &= (\Lambda(q) \circ \Lambda(q^{-1}L_+^{(0)}q) \circ \Lambda(q^{-1}))_+ + (\Lambda(q)\Delta(q^{-1}L_+^{(0)}q) \circ \Delta^{-1} \circ \Lambda(q^{-1}))_+ \\ &= \Lambda(L_+^{(0)}) + \Lambda(q)\Delta(q^{-1}L_+^{(0)}q)_{\geq 1}\Delta^{-1} \circ \Lambda(q^{-1}), \end{aligned}$$

where used the identity (2.11).

$$\begin{aligned} L_-^{(1)} &= (T_d(q) \circ L^{(0)} \circ T_d^{-1}(q))_- \\ &= (\Lambda(q)\Delta \circ q^{-1}L_+^{(0)} \circ q \circ \Delta^{-1} \circ \Lambda(q^{-1}))_- \\ &\quad + (\Lambda(q)\Delta \circ q^{-1} \circ \sum_{i=1}^m q_i^{(0)}\Delta^{-1}r_i^{(0)} \circ q \circ \Delta^{-1} \circ \Lambda(q^{-1}))_-. \end{aligned}$$

Where

$$\begin{aligned} &(\Lambda(q)\Delta \circ q^{-1}L_+^{(0)} \circ q \circ \Delta^{-1} \circ \Lambda(q^{-1}))_- \\ &= (T_d(q)L_+^{(0)} \circ q)_0 \circ \Delta^{-1} \circ \Lambda(q^{-1}) \\ &= T_d(q)L_+^{(0)}(q) \circ \Delta^{-1} \circ \Lambda(q^{-1}), \end{aligned}$$

and

$$\begin{aligned}
& (\Lambda(q)\Delta \circ q^{-1} \circ \sum_{i=1}^m q_i^{(0)} \Delta^{-1} r_i^{(0)} \circ q \circ \Delta^{-1} \circ \Lambda(q^{-1}))_- \\
& \stackrel{(2.10)}{=} \sum_{i=1}^m T_d(q) \circ q_i^{(0)} \cdot \Delta^{-1}(r_i^{(0)} q) \Delta^{-1} \circ \Lambda(q^{-1}) \\
& - \sum_{i=1}^m T_d(q) \circ q_i^{(0)} \Delta^{-1} \circ \Lambda(\Delta^{-1}(r_i^{(0)} q)) \cdot \Lambda(q^{-1}) \\
& = \sum_{i=1}^m T_d(q)(q_i^{(0)} \Delta^{-1} r_i^{(0)})(q) \Delta^{-1} \circ \Lambda(q^{-1}) + \sum_{i=1}^m T_d(q)(q_i^{(0)} \Delta^{-1} \circ (\Delta^{-1})^*(r_i^{(0)} q) \cdot \Lambda(q^{-1})).
\end{aligned}$$

When the above two formulas are substituted into  $L_-^{(1)}$ , then

$$\begin{aligned}
L_-^{(1)} &= T_d(q)L_+^{(0)}(q) \circ \Delta^{-1} \circ \Lambda(q^{-1}) + \sum_{i=1}^m T_d(q)(q_i^{(0)} \Delta^{-1} r_i^{(0)})(q) \Delta^{-1} \circ \Lambda(q^{-1}) \\
&+ \sum_{i=1}^m T_d(q)(q_i^{(0)}) \Delta^{-1} \circ (\Delta^{-1})^*(r_i^{(0)} q) \cdot \Lambda(q^{-1}) \\
&= T_d(q)(L_+^{(0)} + \sum_{i=1}^m q_i^{(0)} \Delta^{-1} r_i^{(0)})(q) \circ \Delta^{-1} \circ \Lambda(q^{-1}) \\
&+ \sum_{i=1}^m T_d(q)(q_i^{(0)}) \Delta^{-1} \circ \Lambda(q^{-1})(\Delta^{-1})^*(qr_i^{(0)}) \\
&= T_d(q)L^{(0)}(q) \circ \Delta^{-1} \circ \Lambda(q^{-1}) + \sum_{i=1}^m T_d(q)(q_i^{(0)}) \Delta^{-1} \circ (T_d^{-1}(q))^*(r_i^{(0)}).
\end{aligned}$$

If let  $q_0^{(1)} = T_d(q)(L^{(0)})(q)$ ,  $r_0^{(1)} = \Lambda(q^{-1})$ ,  $q_i^{(1)} = T_d(q)q_i^{(0)}$ ,  $r_i^{(1)} = (T_d^{-1})^*(q)(r_i^{(0)})$ , we can get this theorem.  $\square$

**Theorem 3.4.** *Under the type 2 gauge transformation  $T_i(r)$ , the transformed Lax operator reads*

$$L^{(1)} = L_+^{(1)} + L_-^{(1)}, \quad (3.12)$$

$$L_+^{(1)} = \Lambda^{-1}(L_+^{(0)}) - \Lambda^{-1}(r^{-1})\Delta^{-1} \circ \Delta^*(r \circ L_+^{(0)} \circ r^{-1})_{\geq 1} \circ \Lambda(r^{-1}), \quad (3.13)$$

$$L_-^{(1)} = q_0^{(1)} \Delta^{-1} r_0^{(1)} + \sum_{i=1}^m q_i^{(1)} \Delta^{-1} r_i^{(1)}, \quad (3.14)$$

$$q_0^{(1)} = \Lambda^{-1}(r^{-1}), r_0^{(1)} = (T_i^{-1}(r))^*(L^{(0)})^*(r), \quad (3.15)$$

$$q_i^{(1)} = T_i(r)(q_i^{(0)}), r_i^{(1)} = (T_i^{-1}(r))^*(r_i^{(0)}). \quad (3.16)$$

*Proof.*

$$\begin{aligned}
L_+^{(1)} &= (T_i(r) \circ L^{(0)} \circ T_i^{-1}(r))_+ \\
&= (\Lambda^{-1}(r^{-1}) \circ \Delta^{-1} \circ r \circ L_+^{(0)} \circ r^{-1} \circ \Delta \circ \Lambda^{-1}(r))_+ \\
&= (\Lambda^{-1}(r^{-1}) \circ \Lambda^{-1}(rL_+^{(0)}r^{-1}) \circ \Lambda^{-1}(r))_+ - (\Lambda^{-1}(r^{-1}) \circ \Delta^{-1} \circ \Delta(\Lambda^{-1}(rL_+^{(0)}r^{-1})) \circ \Lambda^{-1}(r))_+ \\
&= \Lambda^{-1}(L_+^{(0)}) + (\Lambda^{-1}(r^{-1})\Delta^{-1} \circ \Delta^*(r^{-1} \circ (L_+^{(0)})^* \circ r)^* \circ \Lambda(r^{-1}))_+ \\
&= \Lambda^{-1}(L_+^{(0)}) + \Lambda^{-1}(r^{-1})\Delta^{-1} \circ \Delta^*(r(L_+^{(0)})^* \circ r^{-1})_{\geq 1} \Lambda(r^{-1}),
\end{aligned}$$

where used the identity (2.11).

$$\begin{aligned}
L_-^{(1)} &= [T_i(r) \circ (L_+^{(0)} + \sum_{i=1}^m q_i^{(0)} \Delta^{-1} r_i^{(0)}) \circ T_i^{-1}(r)]_- \\
&= (\Lambda^{-1}(r^{-1})\Delta^{-1} \circ rL_+^{(0)} \circ r^{-1} \circ \Delta \circ \Lambda(r))_- \\
&\quad + (\Lambda^{-1}(r^{-1})\Delta^{-1} \circ r(\sum_{i=1}^m q_i^{(0)} \Delta^{-1} r_i^{(0)})r^{-1} \Delta \circ \Lambda^{-1}(r))_-.
\end{aligned}$$

Where

$$\begin{aligned}
&(\Lambda^{-1}(r^{-1})\Delta^{-1} \circ rL_+^{(0)} \circ r^{-1} \circ \Delta \circ \Lambda^{-1}(r))_- \\
&= (\Lambda^{-1}(r^{-1}) \circ \Lambda^{-1}(rL_+^{(0)}r^{-1}) \circ \Lambda^{-1}(r))_- - (\Lambda^{-1}(r^{-1}) \circ \Delta^{-1} \circ \Delta(\Lambda^{-1}(rL_+^{(0)}r^{-1})) \circ \Lambda^{-1}(r))_- \\
&= (\Lambda^{-1}(L_+^{(0)}))_- - (\Lambda^{-1}(r^{-1}) \circ \Delta^{-1} \circ \Lambda^{-1}(r)\Delta\Lambda^{-1}((rL_+^{(0)}r^{-1})^*)^*)_- \\
&= \Lambda^{-1}(r^{-1}) \circ \Delta^{-1} \circ (T_i^{-1}(r))^*(L_+^{(0)})^*(r),
\end{aligned}$$

and

$$\begin{aligned}
&(\Lambda^{-1}(r^{-1})\Delta^{-1} \circ r(\sum_{i=1}^m q_i^{(0)} \Delta^{-1} r_i^{(0)})r^{-1} \Delta \circ \Lambda^{-1}(r))_- \\
&\stackrel{(2.10)}{=} \sum_{i=1}^m \Lambda^{-1}(r^{-1})\Delta^{-1} \circ \Delta(\Delta^{-1}(rq_i^{(0)}) \cdot \Lambda^{-1}(r_i^{(0)}r^{-1}))\Lambda^{-1}(r) \\
&\quad - \sum_{i=1}^m \Lambda^{-1}(r^{-1})\Delta^{-1}(rq_i^{(0)}) \circ \Delta^{-1} \circ \Delta(\Lambda^{-1}(r_i^{(0)}r))\Lambda^{-1}(r) \\
&= \sum_{i=1}^m T_i(r)(q_i^{(0)}) \cdot \Delta^{-1} \circ (T_i^{-1})^*(r)(r_i^{(0)}) \\
&\quad + \Lambda^{-1}(r^{-1})\Delta^{-1} \circ (T_i^{-1}(r))^*(\sum_{i=1}^m q_i^{(0)} \Delta^{-1} r_i^{(0)})^*(r).
\end{aligned}$$

When the above two formulas are substituted into  $L_-^{(1)}$ , then

$$\begin{aligned} L_-^{(1)} &= \Lambda^{-1}(r^{-1}) \circ \Delta^{-1} \circ (T_i^{-1}(r))^*(L_+^{(0)})^*(r) + \Lambda^{-1}(r^{-1}) \Delta^{-1} \circ (T_i^{-1}(r))^* \left( \sum_{i=1}^m q_i^{(0)} \Delta^{-1} r_i^{(0)} \right)^*(r) \\ &\quad + \sum_{i=1}^m T_i(r)(q_i^{(0)}) \cdot \Delta^{-1} \circ (T_i^{-1})^*(r)(r_i^{(0)}) \\ &= \Lambda^{-1}(r^{-1}) \circ \Delta^{-1} \circ (T_i^{-1}(r))^*(L^{(0)})^*(r) + \sum_{i=1}^m T_i(r)(q_i^{(0)}) \cdot \Delta^{-1} \circ (T_i^{-1})^*(r)(r_i^{(0)}). \end{aligned}$$

If let  $q_0^{(1)} = \Lambda^{-1}(r^{-1})$ ,  $r_0^{(1)} = (T_i^{-1}(r))^*(L^{(0)})^*(r)$ ,  $q_i^{(1)} = T_i(r)(q_i^{(0)})$  and  $r_i^{(1)} = (T_i^{-1}(r))^*(L^{(0)})^*(r)$ , then

$$L_-^{(1)} = q_0^{(1)} \Delta^{-1} r_0^{(1)} + \sum_{i=1}^m q_i^{(1)} \Delta^{-1} r_i^{(1)}.$$

□

**Remark:** Although the transformations (3.19) and (3.22) look like as the same as the formula in the semi-discrete KP hierarchy, but there are main difference between the dKP hierarchy and the cdKP hierarchy. We can see the number of the (adjoint) eigenfunctions has added one after each time of gauge transformation. So for the cdKP hierarchy, to ensure that the gauge transformed Lax operator preserves the (2.13), it can be  $q_i^{(1)} \Delta^{-1} r_i^{(1)} = 0$  for some one  $i$ . So the generating function  $q, r$  of the gauge transformation operator  $T_d(q)$  and  $T_i(r)$  can not be arbitrarily chosen. This theorem means there are two choices to keep the form of the Lax operator of cdKP hierarchy. They must be selected from the eigenfunction  $q_i$  and the adjoint eigenfunction  $r_i$  respectively. And the operator  $T_d(q_i)$  and  $T_i(r_i)$  will annihilate their generation functions, i.e. ,

$$T_d(q_i)(q_i) = 0, (T_i^{-1}(r_i))^*(r_i) = 0, i = 1, 2, \dots, m.$$

If the generating function of the gauge transformation in theorem 3.3 was selected for  $q_1$ , then  $q_1^{(1)} = T_d(q_1)(q_1) = 0$ . And  $q_0^{(1)}$  takes over its role.

**Theorem 3.5.** (a). Under the gauge transformation  $L^{(1)} = T_d(q_1) \circ L^{(0)} \circ T_d^{-1}(q_1)$ , the eigenfunction  $q_i^{(0)} = q_i$  and adjoint eigenfunction  $r_i^{(0)} = r_i$  of  $L^{(0)} = L$  are transformed into new eigenfunction  $q_i^{(1)}$  and new adjoint eigenfunction  $r_i^{(1)}$  of  $L^{(1)}$  by

$$q_1^{(1)} = T_d(q_1^{(0)})(L^{(0)})(q_1), r_1^{(1)} = \Lambda(q_1^{-1}), \quad (3.17)$$

$$q_i^{(1)} = T_d(q_1)q_i^{(0)}, r_i^{(1)} = (T_d^{-1})^*(q_1)(r_i^{(0)}), i = 2, \dots, m, \quad (3.18)$$

and the  $\tau$  function  $\tau_{\Delta}^{(0)}$  of  $L^{(0)}$  is transformed into the new  $\tau$  function  $\tau_{\Delta}^{(1)}$  of  $L^{(1)}$  by

$$\tau_{\Delta}^{(1)} = q_1 \tau_{\Delta}^{(0)}. \quad (3.19)$$



(b). Under the gauge transformation  $L^{(1)} = T_i(r_1) \circ L^{(0)} \circ T_i^{-1}(r_1)$ , the eigenfunction  $q_i^{(0)} = q_i$  and adjoint eigenfunction  $r_i^{(0)} = r_i$  of  $L^{(0)} = L$  are transformed into new eigenfunction  $q_i^{(1)}$  and new adjoint eigenfunction  $r_i^{(1)}$  of  $L^{(1)}$  by

$$q_1^{(1)} = \Lambda^{-1}(r_1^{-1}), r_1^{(1)} = (T_i^{-1}(r_1))^*(L^{(0)})^*(r_1), \quad (3.20)$$

$$q_i^{(1)} = T_i(r)q_i^{(0)}, r_i^{(1)} = (T_i^{-1})^*(r)(r_i^{(0)}), i = 2, \dots, m, \quad (3.21)$$

and the  $\tau$  function  $\tau_{\Delta}^{(0)}$  of  $L^{(0)}$  is transformed into the new  $\tau$  function  $\tau_{\Delta}^{(1)}$  of  $L^{(1)}$  by

$$\tau_{\Delta}^{(1)} = \Lambda^{-1}(r_1)\tau_{\Delta}^{(0)}. \quad (3.22)$$

#### 4. SUCCESSIVE APPLICATIONS OF GAUGE TRANSFORMATIONS

In order to investigate the new result of successive transformations by using the gauge transformation operators, we will discuss successive applications of the difference gauge transformation operator  $T_d$ , which is like to the classical case [20, 22]. Firstly, we only consider the chain of gauge transformation operator of single-channel [20] difference type  $T_d(q_1)$  starting from the initial Lax operator  $L^{(0)} = L$ ,

$$L^{(0)} \xrightarrow{T_d^{(1)}(q_1^{(0)})} L^{(1)} \xrightarrow{T_d^{(2)}(q_1^{(1)})} L^{(2)} \xrightarrow{T_d^{(3)}(q_1^{(2)})} L^{(3)} \rightarrow \dots \rightarrow L^{(n-1)} \xrightarrow{T_d^{(n)}(q_1^{(n-1)})} L^{(n)}. \quad (4.1)$$

Here the index  $i$  in the gauge transformation operator  $T_d^{(i)}(q_1^{(j-1)})$  means the  $i$ -th gauge transformation, and  $q_1^{(j)}$  (or  $r_1^{(j)}$ ) is transformed by  $j$ -steps gauge transformations from  $q_1$  (or  $r_1$ ),  $L^{(k)}$  is transformed by  $k$ -step gauge transformations from the initial Lax operator  $L$ .

Now we firstly consider successive gauge transformations in (4.1). We define the operator as

$$T_m = T_d^{(m)}(q_1^{(m-1)}) \circ \dots \circ T_d^{(2)}(q_1^{(1)}) \circ T_d^{(1)}(q_1^{(0)}), \quad (4.2)$$

in which

$$q_i^{(j)} = T_d^{(j)}(q_1^{(j-1)}) \circ \dots \circ T_d^{(2)}(q_1^{(1)}) \circ T_d^{(1)}(q_1^{(0)})q_i, i, j = 1, \dots, m; \quad (4.3)$$

$$r_k^{(j)} = ((T_d^{(j)})^{-1})^*(q_1^{(j-1)}) \circ \dots \circ ((T_d^{(2)})^{-1})^*(q_1^{(1)}) \circ ((T_d^{(1)})^{-1})^*(q_1^{(0)})r_k, j, k = 1, \dots, m. \quad (4.4)$$

In order to express the determinant representation of  $T_m$ , we would like to define the generalized discrete  $\Delta$ -Wronskian for the eigenfunctions  $\{q_1, q_2, \dots, q_m\}$  of  $L$  as

$$W_m^{\Delta}(q_1, q_2, \dots, q_m) = \begin{vmatrix} q_1 & q_2 & \dots & q_m \\ \Delta q_1 & \Delta q_2 & \dots & \Delta q_m \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{m-1} q_1 & \Delta^{m-1} q_2 & \dots & \Delta^{m-1} q_m \end{vmatrix}, \quad (4.5)$$

$$IW_{m+1}^{\Delta}(q_1, q_2, \dots, q_m) = \begin{vmatrix} q_1 \circ \Delta^{-1} & \Lambda(q_1) & \Lambda(\Delta q_1) & \dots & \Lambda(\Delta^{m-2} q_1) \\ q_2 \circ \Delta^{-1} & \Lambda(q_2) & \Lambda(\Delta q_2) & \dots & \Lambda(\Delta^{m-2} q_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_m \circ \Delta^{-1} & \Lambda(q_m) & \Lambda(\Delta q_m) & \dots & \Lambda(\Delta^{m-2} q_m) \end{vmatrix}. \quad (4.6)$$

Using the gauge transformation operator  $T_d(q_1)$ , the  $m$ -step gauge transformation can be construct for

$$L^{(0)} \xrightarrow{T_d^{(1)}(q_1^{(0)})} L^{(1)} \xrightarrow{T_d^{(2)}(q_1^{(1)})} L^{(2)} \xrightarrow{T_d^{(3)}(q_1^{(2)})} L^{(3)} \rightarrow \dots \rightarrow L^{(m-1)} \xrightarrow{T_d^{(m)}(q_1^{(m-1)})} L^{(m)}. \quad (4.7)$$

If  $\eta_i$  is defined by

$$\eta_{i+1} \triangleq (L^{(0)})^i \cdot q_1^{(0)}, \quad (4.8)$$

$\eta_i^{(j)}$  is the  $j$ -step transformed form from  $\eta_i$ . It is easy got  $\eta_{i+1}^{(i)} = q_1^{(i)}$ ,  $i = 1, \dots, m$ , by the mathematical induction.

**Theorem 4.1.** [16] *The gauge transformation operator  $T_m$  and  $T_m^{-1}$  have the following determinant representation:*

$$\begin{aligned} T_m &= T_d^{(m)}(\eta_m^{(m-1)}) \circ \dots \circ T_d^{(2)}(\eta_2^{(1)}) \circ T_d^{(1)}(\eta_1^{(0)}) \\ &= \frac{1}{W_m^\Delta(\eta_1, \eta_2, \dots, \eta_m)} \begin{vmatrix} \eta_1 & \eta_2 & \dots & \eta_m & 1 \\ \Delta\eta_1 & \Delta\eta_2 & \dots & \Delta\eta_m & \Delta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Delta^{m-1}\eta_1 & \Delta^{m-1}\eta_2 & \dots & \Delta^{m-1}\eta_m & \Delta^{m-1} \\ \Delta^m\eta_1 & \Delta^m\eta_2 & \dots & \Delta^m\eta_m & \Delta^m \end{vmatrix}, \end{aligned} \quad (4.9)$$

and

$$T_m^{-1} = \begin{vmatrix} \eta_1 \circ \Delta^{-1} & \Lambda(\eta_1) & \Lambda(\Delta\eta_1) & \dots & \Lambda(\Delta^{m-2}\eta_1) \\ \eta_2 \circ \Delta^{-1} & \Lambda(\eta_2) & \Lambda(\Delta\eta_2) & \dots & \Lambda(\Delta^{m-2}\eta_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_m \circ \Delta^{-1} & \Lambda(\eta_m) & \Lambda(\Delta\eta_m) & \dots & \Lambda(\Delta^{m-2}\eta_m) \end{vmatrix} \frac{(-1)^{m-1}}{\Lambda(W_m^\Delta(\eta_1, \eta_2, \dots, \eta_m))}. \quad (4.10)$$

Here the determinant of  $T_m$  is expanded by the last column and collecting all sub-determinants on the left side of the  $\Delta^i$  with the action " $\circ$ ". And  $T_m^{-1}$  is expanded by the first column and all the sub-determinants are on the right side with the action " $\circ$ ".

With this representation, the action of  $T_m$  on an arbitrary function  $q$  is given by the following theorem.

**Theorem 4.2.** *Under the action of  $T_m = T_d^{(m)}(q_1^{(m-1)}) \circ \dots \circ T_d^{(2)}(q_1^{(1)}) \circ T_d^{(1)}(q_1^{(0)})$ , the transformed eigenfunctions and the  $\tau$ -function of the cdKP hierarchy from the arbitrary  $L^{(0)}$  are*

given by

$$q_1^{(m)} = T_m \circ (L^{(0)})^m \cdot q_1^{(0)} = T_m \circ \eta_{m+1} = \frac{W_{m+1}^\Delta(\eta_1, \eta_2, \dots, \eta_m, \eta_{m+1})}{W_m^\Delta(\eta_1, \eta_2, \dots, \eta_m)}, \quad (4.11)$$

$$r_1^{(m)} = \Lambda(q_1^{(m-1)}) = \Lambda((T_{m-1} \cdot \eta_m)^{-1}) = \frac{\Lambda(W_{m-1}^\Delta(\eta_1, \eta_2, \dots, \eta_{m-1}))}{\Lambda(W_m^\Delta(\eta_1, \eta_2, \dots, \eta_m))}, \quad (4.12)$$

$$q_i^{(m)} = T_m \cdot q_i^{(0)} = \frac{W_{m+1}^\Delta(\eta_1, \eta_2, \dots, \eta_m, q_i^{(0)})}{W_m^\Delta(\eta_1, \eta_2, \dots, \eta_m)}, i = 2, \dots, m, \quad (4.13)$$

$$r_i^{(m)} = (T_m^{-1})^*(r_i^{(0)}) = (-1)^m \frac{\Lambda(IW_{m+1}^\Delta(r_i^{(0)}, \eta_1, \eta_2, \dots, \eta_m))}{\Lambda(W_m^\Delta(\eta_1, \eta_2, \dots, \eta_m))}, i = 2, \dots, m, \quad (4.14)$$

and

$$\tau_\Delta^{(m)} = W_m^\Delta(\eta_1, \eta_2, \dots, \eta_m) \cdot \tau_\Delta. \quad (4.15)$$

And  $\eta_i$  is defined by (4.8).

## 5. CONCLUSIONS AND DISCUSSIONS

In this paper, we have provided two types of gauge transformation in Theorem 3.4 and 3.5 of the cdKP hierarchy. The new solutions of the cdKP hierarchy along single-channel of difference type  $T_d$  are expressed explicitly by the determinants in Theorem 4.2. This result also can be apply to the  $k$ -constrained dKP hierarchy. These results gives a powerful tool to construct the explicit solution of the discrete soliton equations in the cdKP hierarchy. Thus we can study the particular effects of the discretization of dynamical variables by comparing specific solutions of the discrete and continue equations. We shall do it in the future. Moreover, the presence of the gauge transformation in the cdKP hierarchy shows again that the discretization used to define the dKP hierarchy is well enough so that the solvability is inherited by it.

Different from the generating functions of the gauge transformation of the dKP hierarchy which can be chosen freely, the ones of the gauge transformations of the cdKP hierarchy only can be selected from the eigenfunctions and the adjoint eigenfunctions in the Lax operator of the cdKP hierarchy, which leads to the main particularity of the gauge transformation of the cdKP hierarchy.

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